Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00041-7

J. Appl. Maths Mechs, Vol. 65, No. 3, pp. 361–370, 2001

0021-8928/01/$—see front matter

THE PROPERTIES OF THE VALUE FUNCTIONAL OF A DIFFERENTIAL GAME WITH HEREDITARY INFORMATION†

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(*Received* 4 *September* 2000)

Using the concept of derivatives in multivalued directions, differential inequalities are presented which express in infinitesimal form the defining stability properties of the value function of a differential game with hereditary information [1–7]. As a corollary, stability criteria are obtained for piecewise ci-smooth functionals [7, 8] and for envelopes of a family of ci-smooth functionals.

Relying on the development of approaches proposed previously [9–12] for control problems for ordinary differential systems, this paper continues the investigations of [7, 13], which were devoted to control problems complicated by aftereffect and formalized as a differential game with hereditary information. It was shown in [7] that the value functional of this game is a generalized, in fact minimax, solution of a Cauchy problem with a condition at the right-hand end for a Hamilton–Jacobi equation with coinvariant derivatives (ci-derivatives) [8, 13]. Minimax solutions were defined in [7, 13] in terms of non-local properties of stability with respect to characteristic differential inclusions with aftereffect. This definition agrees with the intuitive meaning of the problem. It is convenient, for example, in proving the appropriate existence, uniqueness and well-posedness theorems [13]. However, in specific problems, it is not infrequently difficult to verify that the required stability properties hold for a solution found through auxiliary constructions. The differential inequalities derived below for the derivatives of the value functional in multivalued directions, which are essentially equivalent to the aforementioned stability properties, are frequently easier to verify – this is the case for piecewise ci-smooth functionals and for envelopes of a family of ci-smooth functionals.

1. COINVARIANT DERIVATIVES AND DERIVATIVES IN MULTIVALUED DIRECTIONS

We will use the notation $C([t_1, t_2], R^n)$, where $t_1, t_2 \in R$, $t_1 \leqslant t_2$, for the space of real functions $x[\cdot] = x[t_1[\cdot]t_2] : [t_1, t_2] \mapsto R^n$, where $x[t]$ denotes the value of a function $x[\cdot]$ at a point $t \in [t_1, t_2]$ and $x[t'[\cdot]t'']$ its restriction to $[t', t''] \subset [t_1, t_2]$. Let t_*, t_0, $T \in R$, $t_* \leqslant t_0 < T$, $C_* = C([t_*, T], R^n)$ and let G be the set of pairs $g = (t, x[t_*[\cdot]t])$ such that $t \in [t_0, T]$, $x[t_*[\cdot]t] \in C([t_*, t], R^n)$. We define a distance function on G as follows:

$$\rho(g_1, g_2) = \max\{\rho^*(g_1, g_2), \rho^*(g_2, g_1)\} \tag{1.1}$$

where

$$g_1 = (t_1, x^{(1)}[t_*[\cdot]t_1]) \in G, \quad g_2 = (t_2, x^{(2)}[t_*[\cdot]t_2]) \in G$$

$$\rho^*(g_{i+1}, g_{2-i}) = \max_{t_* \leqslant \xi \leqslant t_{i+1}} \min_{t_* \leqslant \eta \leqslant t_{2-i}} \max\{|\xi - \eta|, \ \|x^{(i+1)}[\xi] - x^{(2-i)}[\eta]\|\}, \quad i = 0, 1$$

Throughout this paper, $\|\cdot\|$ will denote the Euclidean norm of a vector.

Everywhere below, continuity properties of the quantities being considered with respect to the argument $g = (t, x[t_*[\cdot]t])$ will be understood with respect to variation of the argument as estimated by the function ρ.

†*Prikl. Mat. Mekh.* Vol. 65, No. 3, pp. 375–384, 2001.

Consider a functional

$$G \ni g = (t, x[t_*[\cdot]t]) \mapsto \varphi(g) = \varphi(t, x[t_*[\cdot]t]) \in R$$

Fix $g = (t, x[t_*[\cdot]t]) \in G$, $t < T$. Let Lip (g) denote the set of functions $y[\cdot] \in C_*$ which are identical with $x[t_*[\cdot]t]$ in $[t_*, t]$, each of which satisfies a Lipschitz condition in $[t, T]$ (each with its "own" Lipschitz constant). We shall say [7, 8, 13] that a functional φ is coinvariantly differentiable at a point g relative to Lip (g) (ci-differentiable at g) if a number $\partial_t\varphi(g)$ and an n-vector $\nabla\varphi(g)$ exist such that, for any function $y[\cdot] \in$ Lip (g),

$$\begin{aligned} &\varphi(t+\delta, y[t_*[\cdot]t+\delta]) - \varphi(t, x[t_*[\cdot]t]) = \\ &= \partial_t\varphi(g)\,\delta + \langle \nabla\varphi(g), y[t+\delta] - x[t]\rangle + o_{y[\cdot]}(\delta), \qquad \delta \in [0, T-t] \end{aligned} \tag{1.2}$$

where $o_{y[\cdot]}(\delta)$ depends on the choice of $y[\cdot] \in$ Lip (g), $o_{y[\cdot]}(\delta)/\delta \to 0$ as $\delta \to +0$.

Throughout, $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors.

The quantities $\partial_t\varphi(g)$ and $\nabla\varphi(g)$ will be called, respectively, the ci-derivative with respect to t and the ci-gradient of the functional φ at the point g. A functional φ is said to be ci-differentiable if it is ci-differentiable at every point $g = (t, x[t_*[\cdot]t]) \in G$, $t < T$. A continuous ci-differentiable functional φ is said to be ci-smooth.

Note that the class of ci-smooth functionals is quite large. For example, many functionals which can be represented in integral form are ci-smooth; moreover, in most cases the ci-derivatives may be calculated by standard means, relying on the rules for the differentiation of ordinary functions of a finite-dimensional argument. More detailed information on the properties, methods of calculation and some applications of ci-derivatives of functionals can be found in [8].

Let $g = (t, x[t_*]t]) \in G$, $t < T$, $y[\cdot] \in$ Lip (g), $\varepsilon > 0$, and let $F \subset R^n$ be a non-empty convex compact set. The symbol $[F]^\varepsilon$ will denote the closed ε-neighbourhood of the set F in R^n. Define

$$\begin{aligned} &\partial^-\varphi(g \mid y[\cdot]) = \lim_{\delta\downarrow 0} \inf [\varphi(t+\delta, y[t_*[\cdot]t+\delta]) - \varphi(g)]\delta^{-1} \\ &\partial^+\varphi(g \mid y[\cdot]) = \lim_{\delta\downarrow 0} \sup [\varphi(t+\delta, y[t_*[\cdot]t+\delta]) - \varphi(g)]\delta^{-1} \end{aligned} \tag{1.3}$$

$$\Omega(g, F, \varepsilon) = \{y[\cdot] \in \mathrm{Lip}(g) : dy[\tau]/d\tau \in [F]^\varepsilon \text{ a.e. } \tau \in [t, T]\} \tag{1.4}$$

$$\begin{aligned} &d^-\varphi(g \mid F) = \lim_{\varepsilon\downarrow 0} \inf_{y[\cdot]\in\Omega(g, F, \varepsilon)} \partial^-\varphi(g \mid y[\cdot]) \\ &d^+\varphi(g \mid F) = \lim_{\varepsilon\downarrow 0} \sup_{y[\cdot]\in\Omega(g, F, \varepsilon)} \partial^+\varphi(g \mid y[\cdot]) \end{aligned} \tag{1.5}$$

The quantities $\partial^-\varphi(g\,|\,y[\cdot])$ and $\partial^+\varphi(g\,|\,y[\cdot])$ are respectively the lower and upper right derived numbers of the functional φ at the point g along the function $y[\cdot]$. The quantities $d^-\varphi(g\,|\,F)$ and $d^+\varphi(g\,|\,F)$ are known respectively as the lower and upper (right) derivatives of the functional φ at the point g with respect to the multivalued direction F. Note that these derivatives may also take the improper values $-\infty$ and $+\infty$.

Proposition 1. If the functional $\varphi : G \mapsto R$ is ci-differentiable at the point g, then for any convex compact set $F \subset R^n$

$$\begin{aligned} &d^-\varphi(g \mid F) = \partial_t\varphi(g) + \min_{f\in F}\langle \nabla\varphi(g), f\rangle \\ &d^+\varphi(g \mid F) = \partial_t\varphi(g) + \max_{f\in F}\langle \nabla\varphi(g), f\rangle \end{aligned} \tag{1.6}$$

Proof. On the one hand, noting (1.2), (1.3) and using the Mean-Value Theorem for vector-valued functions (see, e.g., [14, p. 5]), we see that for any $\varepsilon > 0$ and $y[\cdot] \in \Omega(g, F, \varepsilon)$

$$\partial^-\varphi(g \mid y[\cdot]) \geqslant \partial_t\varphi(g) + \min_{f\in[F]^\varepsilon}\langle \nabla\varphi(g), f\rangle = \partial_t\varphi(g) + \min_{f\in F}\langle \nabla\varphi(g), f\rangle - \|\nabla\varphi(g)\|\varepsilon$$

and therefore, by (1.5), we have

$$d^-\varphi(g\mid F)\geqslant\partial_t\varphi(g)+\min_{f\in F}\langle\nabla\varphi(g),f\rangle \tag{1.7}$$

On the other hand, since for any $\varepsilon>0$ and $f\in F$ the function

$$y_f[\cdot]=\{y_f[\tau]=x[\tau],\ \tau\in[t_*,t];\ \ y_f[\tau]=x[t]+f(\tau-t),\ \tau\in(t,T]\} \tag{1.8}$$

is in the set $\Omega(g,F,\varepsilon)$, we deduce, again using (1.2), that

$$d^-\varphi(g\mid F)\leqslant\inf_{f\in F}\partial^-\varphi(g\mid y_f[\cdot])=\partial_t\varphi(g)+\min_{f\in F}\langle\nabla\varphi(g),f\rangle \tag{1.9}$$

Inequalities (1.7) and (1.9) prove the first of equalities (1.6). The verification of the second equality is similar.

Let

$$\varphi(g)=\min_{i\in I}\max_{j\in J}\psi_{ij}(g),\quad g=(t,x[t_*[\cdot]t])\in G \tag{1.10}$$

where $\psi_{ij}:G\mapsto R$ are ci-smooth functionals ($i\in I, j\in J$), with I and J finite sets.

A functional $\varphi:G\mapsto R$, representable in the form (1.10), will be called piecewise ci-smooth. Put

$$I_0(g)=\{i_0\in I:\max_{j\in J}\psi_{i_0j}(g)=\varphi(g)\}$$

$$J_0(g,i)=\{j_0\in J:\psi_{ij_0}(g)=\max_{j\in J}\psi_{ij}(g)\} \tag{1.11}$$

$$\omega_0(g,\xi,z)=\min_{i\in I_0(g)}\max_{j\in J_0(g,i)}[\partial_t\psi_{ij}(g)\xi+\langle\nabla\psi_{ij}(g),z\rangle] \tag{1.12}$$

Proposition 2. Let $\varphi:G\mapsto R$ be a piecewise ci-smooth functional (1.10). Then for any convex compact set $F\subset R^n$

$$d^-\varphi(g\mid F)=\min_{f\in F}\omega_0(g,1,f),\quad d^+\varphi(g\mid F)=\max_{f\in F}\omega_0(g,1,f) \tag{1.13}$$

$$g=(t,x[t_*[\cdot]t])\in G,\quad t<T$$

Proof. Let $g=(t,x[t_*[\cdot]t])\in G$, $t<T$, $\varepsilon>0$, $y[\cdot]\in\Omega(g,F,\varepsilon)$. Since the functionals ψ_{ij} are continuous and the sets I and J are finite, a number $\delta_0>0$ exists such that, for all $\delta\in(0,\delta_0)$ and $i\in I$,

$$I_0(t+\delta,y[t_*[\cdot]t+\delta])\subset I_0(g),\quad J_0(t+\delta,y[t_*[\cdot]t+\delta],i)\subset J_0(g,i)$$

Taking this into consideration in (1.10), (1.11) and using the ci-differentiability of the functions ψ_{ij}, we obtain

$$\varphi(g)=\psi_{ij}(g),\quad i\in I_0(g),\ j\in J_0(g,i)$$

$$\varphi(t+\delta,y[t_*[\cdot]t+\delta])-\varphi(g)=\min_{i\in I_0(g)}\max_{j\in J_0(g,i)}[\psi_{ij}(t+\delta,y[t_*[\cdot]t+\delta])-\psi_{ij}(g)]=$$

$$=\min_{i\in I_0(g)}\max_{j\in J_0(g,i)}[\partial_t\psi_{ij}(g)\delta+\langle\nabla\psi_{ij}(g),y[t+\delta]-x[t]\rangle+o_{y[\cdot]}(\delta;i,j)],\quad\delta\in(0,\delta_0]$$

Since I and J are finite, a function $o^*_{y[\cdot]}(\delta)$ exists such that $|o_{y[\cdot]}(\delta;i,j)|\leqslant o^*_{y[\cdot]}(\delta)$ for all $i\in I_0(g)$, $j\in J_0(g,i)$ and $\delta\in(0,\delta_0]$, $o^*_{y[\cdot]}(\delta)/\delta\to0$ as $\delta\downarrow0$. We thus conclude that for any $g=(t,x[t_*[\cdot]t])\in G$, $t<T$, $\varepsilon>0$, $y[\cdot]\in\Omega(g,F,\varepsilon)$ and $\delta\in(0,T-t)$ there is an equality

$$\varphi(t+\delta,y[t_*[\cdot]t+\delta])-\varphi(g)=\omega_0(g,\delta,y[t+\delta]-x[t])+o_{y[\cdot]}(\delta) \tag{1.14}$$

Equalities (1.13) follow from (1.3)–(1.5) and (1.12), (1.14) by arguments similar to those adduced above when proving Proposition 1.

Consider the functional

$$\varphi(g) = \max_{l \in L} [\psi(g, l) + \nu(l)], \quad g = (t, x[t_*[\cdot]t]) \in G \tag{1.15}$$

where $L \subset R^m$ is a non-empty compact set, $\nu : L \mapsto R$ is an upper semicontinuous function, $\psi(g, l) : G \times : L \mapsto R$ is a continuous functional (which is moreover equicontinuous with respect to g as a function of $l \in L$) which is ci-differentiable for any fixed $l \in L$, its ci-derivatives $\partial_t\psi(g, l) \in R$ and $\nabla\psi(g, l) \in R^n$ are continuous with respect to l on L for any fixed $g = (t, x[t_*[\cdot]t]) \in G$, $t < T$, in such a way that the "o-small" term in the appropriate equality of type (1.2) is independent of the choice of $l \in L$.

Put

$$L^0(g) = \{l^0 \in L : \varphi(g) = \psi(g, l^0) + \nu(l^0)\} \tag{1.16}$$

$$\lambda^0(g, f) = \max_{l \in L^0(g)} [\partial_t\psi(g, l) + \langle \nabla\psi(g, l), f \rangle] \tag{1.17}$$

The set $L^0(g)$ is non-empty and compact for any $g \in G$.

Proposition 3. Suppose the functional $\varphi : G \mapsto R$ is representable in the form (1.15). Then for any convex compact subset $F \subset R^n$

$$d^-\varphi(g \mid F) = \min_{f \in F} \lambda^0(g, f), \quad d^+\varphi(g \mid F) = \max_{f \in F} \lambda^0(g, f) \tag{1.18}$$
$$g = (t, x[t_*[\cdot]t]) \in G, \ t < T$$

Proof. We will prove the first equality of (1.18). Let $g = (t, x[t_*[\cdot]t]) \in G$, $t < T$, $\varepsilon > 0$, $y[\cdot] \in \Omega(g, F, \varepsilon)$ and let $\delta_j \downarrow 0$ $(j = 1, 2, \ldots)$ be a sequence such that

$$\partial^-\varphi(g \mid y[\cdot]) = \lim_{j \to \infty} [\varphi(t + \delta_j, y[t_*[\cdot]t + \delta_j]) - \varphi(g)]\delta_j^{-1} \tag{1.19}$$

Put $f_j = (y[t + \delta_j] - x[t])\delta_j^{-1}$ $(j = 1, 2, \ldots)$. Since $y[\cdot] \in \Omega(g, F, \varepsilon)$ (see (1.4)) and $F \subset R^n$ is convex and compact, it follows from the Mean-Value Theorem for vector-valued functions that $f_j \in [F]^\varepsilon$; we may assume, without affecting the generality of the subsequent arguments, that $f_j \to f_* \in [F]^\varepsilon$ as $j \to \infty$. Let $l^0 \in L^0(g)$. Then we conclude from (1.19), taking (1.15), (1.16) and the ci-differentiability of the functional $\psi(g, l^0)$ into account, that

$$\partial^-\varphi(g \mid y[\cdot]) \geqslant \lim_{j \to \infty} [\psi(t + \delta_j, y[t_*[\cdot]t + \delta_j], l^0) - \psi(g, l^0)]\delta_j^{-1} =$$
$$= \lim_{j \to \infty} [\partial_t\psi(g, l^0) + \langle \nabla\psi(g, l^0), f_j \rangle + o_{y[\cdot]}(\delta_j)\delta_j^{-1}] = \partial_t\psi(g, l^0) + \langle \nabla\psi(g, l^0), f_* \rangle$$

These relations hold for any $l^0 \in L^0(g)$. Therefore, taking (1.17) into account and noting also that $f_* \in [F]^\varepsilon$, we obtain

$$\partial^-\varphi(g \mid y[\cdot]) \geqslant \lambda^0(g, f_*) \geqslant \min_{f \in [F]^\varepsilon} \lambda^0(g, f) \geqslant \min_{f \in F} \lambda^0(g, f) - \varepsilon \max_{l \in L} \| \nabla\psi(g, l) \|$$

These inequalities hold for any $\varepsilon > 0$ and $y[\cdot] \in \Omega(g, F, \varepsilon)$, so that, by (1.5), we have

$$d^-\varphi(g \mid F) \geqslant \min_{f \in F} \lambda^0(g, f) \tag{1.20}$$

On the other hand, let $f \in F$, $y_f[\cdot]$ (see (1.8)) and the sequence $\delta_i \downarrow 0$ $(i = 1, 2, \ldots)$ be such that

$$\partial^-\varphi(g \mid y_f[\cdot]) = \lim_{i \to \infty} [\varphi(t + \delta_i, y_f[t_*[\cdot]t + \delta_i]) - \varphi(g)]\delta_i^{-1} \tag{1.21}$$

Take $l_i \in L^0(t + \delta_i, y_f[t_*[\cdot]t + \delta_i])$ $(i = 1, 2, \ldots)$. We now deduce from (1.21), again taking (1.15), (1.16) and the properties of the functional $\psi(g, l)$ into account, as well as (1.8), that

$$\partial^- \varphi(g \mid y_f[\cdot]) \leq \lim_{i\to\infty} [\psi(t+\delta_i, y_f[t_*[\cdot]t+\delta_i], l_i) - \psi(g, l_i)]\delta_i^{-1} =$$

$$= \lim_{i\to\infty} [\partial_t \psi(g, l_i) + \langle \nabla\psi(g, l_i), f\rangle + o_{y_f[\cdot]}(\delta_i)\delta_i^{-1}] \tag{1.22}$$

It follows from (1.1) and (1.8) that $\rho((t + \delta_i, y_j\,[t_*[\cdot]t + \delta_i])\, g) \to 0$ as $i \to \infty$. Therefore, by the aforementioned properties of L, $\psi(g, l)$ and $\nu(l)$, it can be shown that a convergent subsequence $\{l_{i'}\}$ of the sequence $\{l_i\}$ exists such that $l_{i'} \to l_0 \in L^0(g)$ as $i' \to \infty$. The following inequalities thus follow from (1.17) and (1.22)

$$\partial^- \varphi(g \mid y_f[\cdot]) \leq \partial_t \psi(g, l_0) + \langle \nabla\psi(g, l_0), f\rangle \leq \lambda^0(g, f)$$

and they hold for any $f \in F$. Since $y_f[\cdot] \in \Omega(g, F, \varepsilon)$ for all $\varepsilon > 0$ and $f \in F$, we conclude, in accordance with (1.5), that

$$d^- \varphi(g \mid F) \leq \inf_{f\in F} \partial^- \varphi(g \mid y_f[\cdot]) \leq \min_{f\in F} \lambda^0(g, f) \tag{1.23}$$

Inequalities (1.20) and (1.23) prove the first of equalities (1.18). The verification of the second proceeds along the same lines, with self-evident modifications.

2. DIFFERENTIAL INEQUALITIES FOR THE VALUE FUNCTIONAL

Consider a differential game with hereditary information [1–7] for a system with aftereffect

$$dx[t]/dt = f(t, x[t_*[\cdot]t], u, v), \qquad t_* \leq t_0 \leq t \leq T$$
$$x \in R^n, \qquad u \in U \subset R^r, \qquad v \in V \subset R^k \tag{2.1}$$

the performance index of the motion being

$$\gamma = \sigma(x[t_0[\cdot]T]) \tag{2.2}$$

Here x is the phase vector, u and v are the controls of the first and second players, respectively, t_*, t_0 and T are given times ($t_0 < T$), U and V are known compact sets, $x[t_*[\cdot]t] \in C([t_*, t], R^n$ is the motion history up to time t, the functional $\sigma: C([t_0, T], R^n) \mapsto R$ is continuous, and the function $f: G \times U \times V \mapsto R^n$ satisfies the following conditions

(1_f) it is continuous;

(2_f) for any compact set $D \subset C_*$ a number $\Lambda > 0$ exists such that, for all $l \in [t_0, T]$, $u \in U$, $v \in V$, and $x'[\cdot] \in D, x''[\cdot] \in D$, the following estimate (the Lipschitz condition with respect to $x[t_*[\cdot]t)$ holds

$$\| f(t, x'[t_*[\cdot]t], u, v) - f(t, x''[t_*[\cdot]t], u, v)\| \leq \Lambda \max_{t_* \leq \tau \leq t} \| x'[\tau] - x''[\tau]\|$$

(3_f) a number $\varkappa > 0$ exists such that, for all $(t, x[t_*[\cdot]t], u, v) \in G \times U \times V$, the following inequality holds

$$\| f(t, x[t_*[\cdot]t], u, v)\| \leq \varkappa(1 + \max_{t_* \leq \tau \leq t} \| x[\tau]\|)$$

(4_f) for any $(t, x[t_*[\cdot]t]) \in G$ and $s \in R^n$, the following equality (the saddle-point condition in a small game [1, 5–7]) holds

$$\min_{u\in U} \max_{v\in V} \langle s, f(t, x[t_*[\cdot]t], u, v)\rangle = \max_{v\in V} \min_{u\in U} \langle s, f(t, x[t_*[\cdot]t], u, v)\rangle$$

Conditions (1_f)–(3_f) guarantee the existence of a unique solution (motion) $x[t_*[\cdot]T]$, extendible up to T, of system (2.1) for any initial state $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G$, $t^0 < T$ (i.e., $x[\tau] = x^0[\tau]$, $t_* \leq \tau \leq t^0$) and Borel-measurable realizations $u = u[t] \in U$ and $v = v[t] \in V$, $t^0 \leq t < T$. The addition of condition (4_f) guarantees that the game is solvable in pure strategies.

The first player's aim is to minimize the index γ (2.2), and that of the second is to maximize γ. By control strategies of the first and second player we mean arbitrary functions $u(t, x[t_*[\cdot]t]) \in U$ and $v(t, x[t_*[\cdot]t]) \in V$, respectively, where $(t, x[t_*[\cdot]t]) \in G$, $t \in [t^0, T]$. Motion takes place on the basis of a selected strategy in a discrete time scheme. A detailed formalization of the game (2.1), (2.2) was given in [7]. It was proved there that the game has a value – a functional from G to R which is the unique minimax [13] solution of a Cauchy problem for the following Hamilton–Jacobi equation with ci-derivatives

$$\partial_t \varphi(g) + H(g, \nabla\varphi(g)) = 0, \quad g = (t, x[t_*[\cdot]t]) \in G, \quad t < T \tag{2.3}$$

where

$$H(g, s) = \max_{v \in V} \min_{u \in U} \langle s, f(g, u, v)\rangle \tag{2.4}$$

with the following condition at the right-hand end of the interval

$$\varphi(T, x[t_*[\cdot]T]) = \sigma(x[t_0[\cdot]T]), \qquad x[\cdot] \in C_* \tag{2.5}$$

The optimal strategies $u^0(t, x[t_*[\cdot]t])$ and $v^0(t, x[t_*[\cdot]t])$ that make up the saddle point of the game are constructed as extremals for this solution.

Let P and Q be certain non-empty sets. Let us consider multivalued mappings

$$G \times Q \ni (g = (t, x[t_*[\cdot]t]), q) \mapsto F^*(g, q) \subset R^n$$
$$G \times P \ni (g = (t, x[t_*[\cdot]t]), p) \mapsto F_*(g, p) \subset R^n$$

satisfying the following conditions

(1_K) for any $(g, p, q) \in G \times P \times Q$, the sets $F^*(g, q)$ and $F_*(g, p)$ are non-empty convex compact sets in R^n; a number $a > 0$ exists such that

$$\max \{\| f \| \;\; | \, f \in F^*(t, x[t_*[\cdot]t], q) \cup F_*(t, x[t_*[\cdot]t], p)\} \leqslant a(1 + \max_{t_* \leqslant \tau \leqslant t} \| x[\tau] \|)$$
$$(t, x[t_*[\cdot]t], p, q) \in G \times P \times Q$$

(2_K) for any $p \in P$ and $q \in Q$ the multivalued mappings $G \ni g \mapsto F^*(g, q)$ and $G \ni g \mapsto F_*(g, p)$ are continuous in the Hausdorff metric;

(3_K) for any $g \in G$ and $s \in R^n$,

$$\sup_{q \in Q} \min_{f \in F^*(g, q)} \langle s, f\rangle = H(g, s) = \inf_{p \in P} \max_{f \in F_*(g, p)} \langle s, f\rangle$$

We will call the pairs $\{Q, F^*(\cdot)\}$ and $\{P, F_*(\cdot)\}$ upper and lower characteristic complexes (CCs), respectively. The set of all upper CCs will be denoted by $K_c^*(H)$, and that of all lower CCs by $K_*^c(H)$. By virtue of (1_f)–(4_f) and (2.4), conditions (1_K)–(3_K) will hold, for example, when

$$Q = V, \quad F^*(g, q) = \text{co}\{f(g, u, q) \,|\, u \in U\}$$
$$P = U, \quad F_*(g, p) = \text{co}\{f(g, p, v) \,|\, v \in V\} \tag{2.6}$$

where co F is the convex hull of the set F in R^n, so that $K_c^*(H) \neq \emptyset$ and $K_*^c(H) \neq \emptyset$.

Let

$$\{Q, F^*(\cdot)\} \in K_c^*(H), \quad \{P, F_*(\cdot)\} \in K_*^c(H)$$

Theorem 1. A necessary and sufficient condition for a functional $\varphi : G \mapsto R$ to be the value functional of a differential game (2.1), (2.2) with hereditary information is that it should be continuous and satisfy the boundary condition (2.5) and the following differential inequalities

$$\sup_{q \in Q} d^- \varphi(g \,|\, F^*(g, q)) \leqslant 0, \quad g = (t, x[t_*[\cdot]t]) \in G, \quad t < T \tag{2.7}$$

$$\inf_{p \in P} d^+ \varphi(g \,|\, F_*(g, p)) \geqslant 0, \quad g = (t, x[t_*[\cdot]t]) \in G, \quad t < T \tag{2.8}$$

Proof. We establish a correspondence between characteristic complexes $\{Q, F^*(\cdot)\}$ and $\{P, F_*(\cdot)\}$ and characteristic differential inclusions with aftereffect

$$dx[t]/dt \in F^*(t, x[t_*[\cdot]t], q) \tag{2.9}$$

$$dx[t]/dt \in F_*(t, x[t_*[\cdot]t], p) \tag{2.10}$$

For fixed $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G$, $q \in Q$ and $p \in P$, let $X^*(g^0, q|F^*(\cdot))$ and $X_*(g^0, p|F_*(\cdot))$ denote the sets of solutions of characteristic differential inclusions (2.9) and (2.10), respectively, issuing from g^0 (that is, the sets of functions of C_* that are identical with $x^0[t_*[\cdot]t^0]$ in $[t_*, t^0]$, are absolutely continuous in $[t^0, T]$, and satisfy inclusions (2.9) and (2.10), respectively, for almost all $t \in [t^0, T]$). By virtue of (1_K), (2_K) (see [14, 15]), these sets are non-empty compact subsets of C_*.

By the results of [7, 13], the theorem will be proved if we can show that differential inequalities (2.7) and (2.8) are equivalent respectively to the following inequalities

$$\sup_{(g^0, t, q)} \min_{x^*[\cdot]} [\varphi(t, x^*[t_*[\cdot]t]) - \varphi(g^0)] \leq 0 \tag{2.11}$$

$$\inf_{(g^0, t, p)} \max_{x_*[\cdot]} [\varphi(t, x_*[t_*[\cdot]t]) - \varphi(g^0)] \geq 0 \tag{2.12}$$

where

$$g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G, \quad t \in [t^0, T], \quad p \in P, \quad q \in Q$$
$$x^*[\cdot] \in X^*(g^0, q|F^*(\cdot)), \quad x_*[\cdot] \in X_*(g^0, p|F_*(\cdot))$$

These inequalities define non-local stability properties of the functional φ with respect to characteristic differential inclusions (2.9) and (2.10).

We will prove that (2.7) and (2.11) are equivalent.

Proof of the implication (2.11) $\Rightarrow$ (2.7). Let $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G$, $t^0 < T$ and $q \in Q$. Put

$$\tau_i^{(m)} = t^0 + i(T - t^0)/m, \quad i = 0, 1, \ldots, m; \quad m = 1, 2, \ldots$$

By condition (2.11), there are functions $x_{(i)}^{(m)}[\cdot]$ such that

$$x_{(0)}^{(m)}[\cdot] \in X^*(g^0, q \mid F^*(\cdot)), \quad x_{(i)}^{(m)}[\cdot] \in X^*(\tau_{i-1}^{(m)}, x_{(i-1)}^{(m)}[t_*[\cdot]\tau_{i-1}^{(m)}], q \mid F^*(\cdot))$$
$$\varphi(\tau_i^{(m)}, x_{(i)}^{(m)}[t_*[\cdot]\tau_i^{(m)}]) \leq \varphi(\tau_{i-1}^{(m)}, x_{(i-1)}^{(m)}[t_*[\cdot]\tau_{i-1}^{(m)}]), \quad i = 1, \ldots, m \tag{2.13}$$

Put $x_{(m)}[\cdot] = x_{(m)}^{(m)}[\cdot]$. It follows from (2.13) that $x_{(m)}[\cdot] \in X^*(g^0, q|F^*(\cdot))$ and

$$\varphi(\tau_i^{(m)}, x_{(m)}[t_*[\cdot]\tau_i^{(m)}]) \leq \varphi(g^0), \quad i = 0, 1, \ldots, m \tag{2.14}$$

Consider the sequence of functions $x_{(m)}[\cdot]$ $(m = 1, 2, \ldots)$. Since $X^*(g^0, q|F^*(\cdot))$ is a compact subset of C_*, we may assume that this sequence is uniformly convergent to a function $x^*[\cdot] \in X^*(g^0, q|F^*(\cdot))$. Let $t \in [t^0, T]$, $g^* = (t, x^*[t_*[\cdot]t])$, $\tau_m = \tau_m(t) = \max\{\tau_j^{(m)} | \tau_j^{(m)} \leq t\}$, $g_m = (\tau_m, x_{(m)}[t_*[\cdot]\tau_m])$. Then (see (1.1)) $\rho(g_m, g^*) \to 0$ as $m \to \infty$ and (see (2.14)) $\varphi(g_m) \leq \varphi(g^0)$ $(m = 1, 2, \ldots)$. Since φ is a continuous functional, this implies the inequality

$$\varphi(t, x^*[t_*[\cdot]t]) \leq \varphi(g^0), \quad t \in [t^0, T]$$

Since the multivalued mapping $g \mapsto F^*(g, q)$ is continuous and the function $x^*[\cdot]$ is Lipschitzian in $[t^0, T]$, it follows that for any $\varepsilon > 0$ a $\delta_0(\varepsilon) \in (0, T - t^0]$ exists such that, for all $t \in [t^0, t^0 + \delta_0(\varepsilon)]$, we have the inclusion relation $F^*(t, x^*[t_*[\cdot]t], q) \subset [F^*(g^0, q)]^\varepsilon$. Take $y_\varepsilon[\cdot] \in \Omega(t^0 + \delta_0(\varepsilon), x^*[t_*[\cdot]t^0 + \delta_0(\varepsilon)], F^*(g^0, q), \varepsilon)$. Then, by (1.4), the aforementioned properties of the function $x^*[\cdot]$ and the choice of $\delta_0(\varepsilon)$, we have

$$y_\varepsilon[\cdot] \in \Omega(g^0, F^*(g^0, q), \varepsilon), \quad [\varphi(t^0 + \delta, y_\varepsilon[t_*[\cdot]t^0 + \delta]) - \varphi(g^0)]\delta^{-1} \leq 0$$
$$\delta \in (0, \delta_0(\varepsilon)], \quad \varepsilon > 0$$

whence we conclude, by (1.3) and (1.5), that

$$d^{-}\varphi(g^0 \mid F^*(g^0, q)) \leqslant \lim_{\varepsilon\downarrow 0} \sup \partial^{-}\varphi(g^0 \mid y_\varepsilon[\cdot]) \leqslant 0$$

This completes the proof of the implication (2.11) ⇒ (2.7).

Proof of the implication (2.7) ⇒ (2.11). To prove this implication it will suffice to show that (2.7) implies the inequality

$$\sup_{(g^0, t, q, \mu)} \min_{x[\cdot]} [\varphi(t, x[t_*[\cdot]t]) - \varphi(g^0)] \leqslant 0 \tag{2.15}$$

$$(g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G, \quad t \in [t^0, T], \quad q \in Q, \quad \mu > 0, \quad x[\cdot] \in X^*_\mu(g^0, q \mid F^*(\cdot)))$$

where $X^*_\mu(g^0, q|F^*(\cdot))$ is the set of solutions, issuing from g^0, of the differential inclusion $dx[t]/dt \in [F^*(t, x[t_*[\cdot]t], q)]^\mu$.

Indeed, if inequality (2.15) holds, then for any $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G$, $t^0 < T$, $t \in [t^0, T]$, $q \in Q$, and a sequence $\mu_i \downarrow 0$ $(i = 1, 2, \ldots)$, a sequence of functions $x^*_{(i)}[\cdot] \in X^*_{\mu_i}(g^0, q|F^*(\cdot))$ exists such that

$$\varphi(t, x^*_{(i)}[t_*[\cdot]t]) \leqslant \varphi(g^0) \quad (i = 1, 2, \ldots) \tag{2.16}$$

Since the multivalued mapping $F^*(\cdot)$ satisfies conditions (1_K) and (2_K), we can show that this sequence has a uniformly convergent subsequence, whose limit is $x^*_0[\cdot]\, X^*(g^0, q|F^*(\cdot))$. By (2.16), the function $x^*_0[\cdot]$ will satisfy the inequality

$$\varphi(t, x^*_0[t_*[\cdot]t]) \leqslant \varphi(g^0)$$

Thus, we conclude that (2.11) follows from (2.15).

To prove inequality (2.15) we assume the opposite, that is, $\alpha > 0$, $\mu > 0$, $t^0 < T$, $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G$, $t^* \in (t^0, T]$, and $q \in Q$ exist such that

$$\min_{x[\cdot] \in X^*_\mu(g^0, q|F^*(\cdot))} \varphi(t^*, x[t_*[\cdot]t^*]) > \varphi(g^0) + \alpha \tag{2.17}$$

Put

$$\beta(\tau) = \varphi(g^0) + \alpha(\tau - t^0)/(t^* - t^0), \tag{2.18}$$

$$\tau_0 = \sup\{\tau \in [t_0, t^*] \mid \min_{x[\cdot] \in X^*_\mu(g^0, q|F^*(\cdot))} \varphi(\tau, x[t_*[\cdot]\tau]) \leqslant \beta(\tau)\}$$

Since $\beta(t^0) = \varphi(g^0)$, it follows that $\tau_0 \geqslant t^0$. It follows from (2.17) that $\tau_0 < t^*$. The functional $\varphi : G \mapsto R$ is continuous and the set $X^*_\mu(g^0, q|F^*(\cdot))$ is a non-empty compact subset of C_*. Therefore, the supremum in (2.18) is achieved and a function $x_0[\cdot]$ exists such that

$$x_0[\cdot] \in X^*_\mu(g^0, q \mid F^*(\cdot)) \tag{2.19}$$

$$\varphi(\tau_0, x_0[t_*[\cdot]\tau_0]) \leqslant \beta(\tau_0) \tag{2.20}$$

Put $g_0 = (\tau_0, x_0[t_*[\cdot]\tau_0])$. By condition (2.7), we have

$$d^{-}\varphi(g_0|F^*(g_0, q)) \leqslant 0 \tag{2.21}$$

Taking into consideration the continuity of the multivalued mapping $g \mapsto F^*(g, q)$ and definition (1.5) of the lower derivative in a multivalued direction, we deduce from (2.21) that

$$0 < \varepsilon \leqslant \mu/2, \quad y[\cdot] \in \Omega(g_0, F^*(g_0, q), \varepsilon), \quad 0 < \delta < t^* - \tau_0 \tag{2.22}$$

exist for which the following relations hold

$$F^*(g_0, q) \subset [F^*(t, y[t_*[\cdot]t], q)]^{\mu/2}, \qquad t \in [\tau_0, \tau_0 + \delta] \tag{2.23}$$

$$\varphi(\tau_0 + \delta, y[t_*[\cdot]\tau_0 + \delta]) - \varphi(g_0) \leq \alpha\delta/(t^* - t^0) \tag{2.24}$$

Let $x'[\cdot] \in X^*_\mu(\tau_0 + \delta, y[t_*[\cdot]\tau_0 + \delta], q|F^*(\cdot))$. It then follows from (2.19), (2.22) and (2.23), since $\tau_0 \in [t^0, t^*)$, that

$$x'[\cdot] \in X^*_\mu(g^0, q \mid F^*(\cdot)) \tag{2.25}$$

and, taking into consideration that $x'[t_*[\cdot]\tau_0 + \delta] = y[t_*[\cdot]\tau_0 + \delta]$, we obtain from (2.18), (2.20) and (2.24)

$$\varphi(\tau_0 + \delta, x'[t_*[\cdot]\tau_0 + \delta]) \leq \beta(\tau_0 + \delta) \tag{2.26}$$

Since $\delta > 0$, relations (2.25) and (2.26) contradict definition (2.18) of the number τ_0. This contradiction completes the proof of inequality (2.15) and hence also that of the implication (2.7) $\Rightarrow$ (2.11).

Thus, differential inequality (2.7) is equivalent to the stability property (2.11). The proof that (2.8) and (2.12) are equivalent is analogous. The theorem is proved.

We will discuss a few special cases. We first observe that at points g where the functional φ is ci-differentiable, the pair of inequalities (2.7) and (2.8) is equivalent to Eq. (2.3). This follows from Proposition 1 and condition (3_K) imposed on the characteristic complexes $\{Q, F^*(\cdot)\} \in K^*_c(H)$ and $\{P, F_*(\cdot)\} \in K^c_*(H)$. Thus, at points where it is ci-differentiable, the value functional satisfies Eq. (2.3). Together with boundary condition (2.5), this, in general, only necessary property is sufficient in case the value is ci-smooth. However, it is extremely rare that the value functional is ci-smooth. Most often, it turns out to be piecewise ci-smooth or expressible in the form (1.15) (see, e.g., [3, 5–7]). The following corollary of Theorem 1 and Propositions 2, 3 gives stability criteria for such functionals which are fairly easy to verify.

Corollary 1. A piecewise ci-smooth functional (1.10) (respectively, a functional expressible in the form (1.15)) is the value functional of a differential game (2.1), (2.2) if and only if it satisfies condition (2.5) and, for all $g = t, x[t_*[\cdot]t]) \in G, t < T$, the inequalities

$$\sup_{q \in Q} \min_{f \in F^*(g,q)} \chi(g, f) \leq 0 \leq \inf_{p \in P} \max_{f \in F_*(g,p)} \chi(g, f)$$

in which $\chi(g,f) = \omega_0(g, 1, f)$ as defined in (1.12) (respectively, in which $\chi(g,f) = \lambda^0(g,f)$ as defined in (1.17)).

Theorem 1 and Corollary 1 are true for any characteristic complexes $\{Q, F^*(\cdot)\} \in K^*_c(H)$, $\{P, F_*(\cdot)\} \in K^c_*(H)$. In particular, one can always take characteristic complex (2.6). Note that then inequalities (2.11) and (2.12), and therefore also (see the proof of Theorem 1) differential inequalities (2.7) and (2.8), define the properties of u- and v-stability [1–5], respectively, for the value functional of the above differential game. Another universal way to select a characteristic complex is indicated in [7, 13]. We observe, furthermore, that in specific problems suitable selection of a characteristic complex will sometimes considerably simplify verification of the criteria given in Theorem 1 and Corollary 1 for the stability of functionals.

This research was supported financially by the Russian Foundation for Basic Research (00-01-00367).

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Translated by D.L.